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The Wronskian structure for solution of the k-constrained KP hierarchy

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Abstract. In this paper we study a class of $(1+1)$-dimensional integrable systems called the k-constrained KP hierarchy. We first show by using the Wronskian technique that the first flow of the 2-constrained KP hierarchy possesses solutions in generalized double Wronskian form; then we give a conjecture on the Wronskian structure for solutions of the general k-constrained KP hierarchy.

1. Introduction

It is well known that the KP hierarchy constructed in terms of the Sato theory can be reduced to obtain the l-reduced KP hierarchy, which consists of many interesting $(1+1)$-dimensional soliton equations [1]. For example the KdV equation and the Boussinesq equation belong to the 2-reduced and 3-reduced KP hierarchy respectively [1]. There is, however, another important type of reduction for the KP hierarchy, which was given in [2–4]. By using this new type of reduction, one obtains the following k-constrained KP hierarchy [4]:

$$L_{t_n} = [B_n, L] \tag{1.1a}$$

$$q_{t_n} = B_n q \tag{1.1b}$$

$$r_{t_n} = -B_n^* r \qquad n \geqslant 1 \tag{1.1c}$$

where

$$L = \partial + u_2\partial^{-1} + u_3\partial^{-2} + \cdots \tag{1.2}$$

is a micro-differential operator which satisfies the constraint

$$L^k = (L^k)_+ + q\partial^{-1} \circ r \qquad k \geqslant 1 \tag{1.3}$$

while $\partial = \frac{\partial}{\partial x}$, $u_j (j = 2, 3, \cdots)$ and q, r are functions of $t = (t_1, t_2, \cdots)$ with $t_1 = x$, $B_n = (L^n)_+$ is the differential part of the operator L^n, B_n^* is the adjoint of B_n, and $\partial^{-1} \circ r$ is defined by

$$\partial^{-1} \circ r = r\partial^{-1} - r_x\partial^{-2} + r_{xx}\partial^{-3} - \cdots. \tag{1.4}$$

When $q = r = 0$, the k-constrained KP hierarchy (1.1) leads to the usual k-reduced KP hierarchy given in [1]. For general q and r the solutions of the k-constrained KP hierarchy (1.1) provide a submanifold of the solutions of the KP hierarchy [2–4].

The 1-constrained KP hierarchy is just the AKNS hierarchy with the following first flow:

$$q_{t_2} = q_{xx} + 2q^2 r \tag{1.5a}$$

$$r_{t_2} = -r_{xx} - 2qr^2. \tag{1.5b}$$

In the 2-constructed KP hierarchy, the first flow is

$$q_{t_2} = q_{xx} + 2u_2 q \tag{1.6a}$$

$$u_{2,t_2} = (qr)_x \tag{1.6b}$$

$$r_{t_2} = -r_{xx} - 2u_2 r. \tag{1.6c}$$

Equation (1.6) was given by Yajima and Oikawa and was solved by using the inverse scattering method [5]; it has some physical applications. We call it in this paper the Yajima–Oikawa equation.

References [4, 6] showed the Lax pairs, the recursion operators and the bi-Hamiltonian structures of the k-constrained KP hierarchy (1.1). References [7, 8] further showed that the k-constrained KP hierarchy (1.1) can be put into bilinear equations; their rational and soliton solutions can be constructed by using the free Fermion operators. In this paper, we investigate the Wronskian structure for the solutions of the k-constrained KP hierarchy. In section 2, we introduce some useful notation and identities; in section 3, we prove that the Yajima–Oikawa equation (1.6) has solutions in generalized double Wronskian form. Then in section 4, based on the result of section 3, the double Wronskian solution of the AKNS hierarchy and the double-soliton solution of the k-constrained KP hierarchy (1.1), we shall give a conjecture on the Wronskian structure for the solutions of the k-constrained KP hierarchy.

2. Some notation and identities

In this section, we introduce some notation and identities which will be useful for our investigation of the Wronskian structure for the solutions of the k-constrained KP hierarchy.

Let φ and ψ be two column vectors which depend on the variable $t = (t_1, t_2, \cdots)$; $N, M, i_1, \cdots, i_\mu, j_1, \cdots, j_\nu$ be positive integers. Let $\tilde{t} = (\tilde{t}_1, \tilde{t}_2, \cdots) = (\frac{1}{2}t_1, \frac{1}{2}t_2, \cdots)$.
Define

$$|\hat{N}, i_1, \cdots, i_\mu;\ \hat{M}, j_1, \cdots, j_\nu| = \det(\hat{N}, i_1, \cdots, i_\mu;\ \hat{M}, j_1, \cdots, j_\nu) \tag{2.1}$$

where we have asssumed that the above determinant is well defined, and the above matrices are defined as follows:

$$\hat{N} = \left(\varphi, \varphi^{(2)}, \cdots, \varphi^{(2N)}\right) \qquad \hat{M} = \left(\psi, \psi^{(1)}, \cdots, \psi^{(M)}\right)$$

$$\left(\hat{N}, i_1, \cdots, i_\mu;\ \hat{M}, j_1, \cdots, j_\nu\right) = \left(\varphi, \varphi^{(2)}, \cdots, \varphi^{(2N)}, \varphi^{(2i_1)}, \cdots, \varphi^{(2i_\mu)}, \psi, \psi^{(1)}, \cdots, \psi^{(M)}, \psi^{(j_1)}, \cdots, \psi^{(j_\nu)}\right) \tag{2.2}$$

$\varphi^{(i)}$ and $\psi^{(j)}$ denote the column vectors obtained by taking the ith and jth order derivatives of the components of φ and ψ with respect to $\tilde{t}_1$. Similarly we define

$$|\hat{N}_k, i_1, \cdots, i_\mu;\ \hat{M}, j_1, \cdots, j_\nu| = \det\left(\hat{N}_k, i_1, \cdots, i_\mu;\ \hat{M}, j_1, \cdots, j_\nu\right) \tag{2.3}$$

$$|\hat{N}_{kl}, i_1, \cdots, i_\mu;\ \hat{M}, j_1, \cdots, j_\nu| = \det\left(\hat{N}_{kl}, i_1, \cdots, i_\mu;\ \hat{M}, j_1, \cdots, j_\nu\right) \tag{2.4}$$

$$|\hat{N}, i_1, \cdots, i_\mu;\ \hat{M}_k, j_1, \cdots, j_\nu| = \det\left(\hat{N}, i_1, \cdots, i_\mu;\ \hat{M}_k, j_1, \cdots, j_\nu\right) \tag{2.5}$$

and so on, where

$$\hat{N}_k = \left(\varphi, \varphi^{(2)}, \cdots, \varphi^{(2k-2)}, \varphi^{(2k+2)}, \cdots, \varphi^{(2N)}\right)$$
$$\hat{N}_{kl} = \left(\varphi, \varphi^{(2)}, \cdots, \varphi^{(2k-2)}, \varphi^{(2k+2)}, \cdots, \varphi^{(2l-2)}, \varphi^{(2l+2)}, \cdots, \varphi^{(2N)}\right) \tag{2.6}$$
$$\hat{M}_k = \left(\psi, \psi^{(1)}, \cdots, \psi^{(k-1)}, \psi^{(k+1)}, \cdots, \psi^{(M)}\right).$$

The above notation is similar to that given in [9, 10]. In this paper, the determinants of the form (2.1)–(2.5) are said to be in generalized double Wronskian form.

From [9] we have the following identities:

$$\sum_{i=1}^{n} |\nu_1, \cdots, \lambda\nu_i, \cdots, \nu_n| = \left(\sum_{i=1}^{n} \lambda_i\right) |\nu_1, \nu_2, \cdots, \nu_n| \tag{2.7}$$
$$|Eab||Ebc| - |Ebd||Eac| + |Ecd||Eab| = 0 \tag{2.8}$$

where ν_i, a, b, c, d are $n \times 1$ matrices, E is a $n \times (n-2)$ matrix, and $\nu_i = (\nu_i^1, \nu_i^2, \cdots, \nu_i^n)^T, \lambda\nu_i = (\lambda_1\nu_i^1, \lambda_2\nu_i^2, \cdots, \lambda_n\nu_i^n)^T$.

The identities (2.7) and (2.8) are very useful identities in the construction of solutions for some soliton equations by using the Wronskian technique [9].

3. The Wronskian structure for the solutions of the Yajima–Oikawa equations (1.6)

Let us define the functions τ, ρ, σ as follows:

$$u_2 = \frac{\partial^2 \ln\tau}{\partial t_1^2} \qquad q = \frac{\rho}{\tau} \qquad r = \frac{\sigma}{\tau}. \tag{3.1}$$

Then the Yajima–Oikawa equation (1.6) is transformed to the following bilinear equations:

$$D_1 D_2 \tau \circ \tau = 2\rho\sigma \tag{3.2a}$$
$$(D_2 - D_1^2)\rho \circ \tau = 0 \tag{3.2b}$$
$$(D_2 + D_1^2)\sigma \circ \tau = 0 \tag{3.2c}$$

where $D_i, i = 1, 2$ is the Hirota's bilinear operator with respect to t_i. We have the following theorem:

Theorem. Equation (3.2) has the following solution:

$$\tau = |\hat{N};\ \hat{M}| = \det\left(\varphi, \varphi^{(2)}, \cdots, \varphi^{(2N)}, \psi, \psi^{(1)}, \cdots, \psi^{(M)}\right) \tag{3.3a}$$
$$\rho = |\widehat{N+1};\ \widehat{M-1}| = \det\left(\varphi, \varphi^{(2)}, \cdots, \varphi^{(2N+2)}, \psi, \psi^{(1)}, \cdots, \psi^{(M-1)}\right) \tag{3.3b}$$
$$\sigma = |\widehat{N-1};\ \widehat{M+1}| = \det\left(\varphi, \varphi^{(2)}, \cdots, \varphi^{(2N-2)}, \psi, \psi^{(1)}, \cdots, \psi^{(M+1)}\right) \tag{3.3c}$$

where $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_{N+M+2})^T$, $\psi = (\psi_1, \psi_2, \cdots, \psi_{N+M+2})^T$, $\varphi_i = a_i e^{\lambda_i \bar{t}_1 + \lambda_i^2 \bar{t}_2}$, $\psi_i = b_i e^{-\lambda_i \bar{t}_1 - \lambda_i^2 \bar{t}_2}$, and λ_i, a_i, b_i are arbitrary constants, $N, M \geqslant 1$.

Proof. We first prove that τ, ρ, σ defined by (3.3) satisfy equation (3.2a), or equivalently, the following equation:

$$\tau_{\bar{t}_1\bar{t}_2}\tau - \tau_{\bar{t}_1}\tau_{\bar{t}_2} - 4\rho\sigma = 0. \tag{3.4}$$

From the definition we have

$$\begin{aligned}
\tau_{\bar{t}_1} &= \sum_{j=0}^{N}(-1)^{N+j}|\hat{N}_j, 2j+1;\ \hat{M}| + |\hat{N};\ \widehat{M-1}, M+1| \\
\tau_{\bar{t}_2} &= |\widehat{N-1}, 2N+2;\ \hat{M}| - |\hat{N};\ \widehat{M-2}, M+1, M| - |\hat{N};\ \widehat{M-1}, M+2| \\
\tau_{\bar{t}_1\bar{t}_2} &= \sum_{j=0}^{N-1}(-1)^{N-1+j}|(\widehat{N-1})_j, 2j+1, 2N+2;\ \hat{M}| + |\widehat{N-1}, 2N+3;\ \hat{M}| \\
&\quad + |\widehat{N-1}, 2N+2;\ \widehat{M-1}, M+1| - \sum_{j=0}^{N}(-1)^{N+j}|\hat{N}_j, 2j+1;\ \widehat{M-2}, M+1, M| \\
&\quad - |\hat{N};\ \widehat{M-3}, M-1, M+1, M| - |\hat{N};\ \widehat{M-2}, M+2, M| - |\hat{N};\ \widehat{M-1}, M+3| \\
&\quad - \sum_{j=0}^{N}(-1)^{N+j}|\hat{N}_j, 2j+1;\ \widehat{M-1}, M+2| - |\hat{N};\ \widehat{M-2}, M, M+2|.
\end{aligned}$$

By using the identity (2.7) we have

$$\begin{aligned}
&\left(\sum_{i=1}^{N+M+2}\lambda_i\right)\tau = \tau_{\bar{t}_1} - 2|\hat{N};\ \widehat{M-1}, M+1| \\
&\left(\sum_{i=0}^{N+M+2}\lambda_i^2\right)\tau = \tau_{\bar{t}_2} + 2|\hat{N};\ \widehat{M-2}, M+1, M| + 2|\hat{N};\ \widehat{M-1}, M+2| \\
&\left(\sum_{i=0}^{N+M+2}\lambda_i\right)\left(\sum_{i=0}^{N+M+2}\lambda_i^2\right)\tau = \tau_{\bar{t}_1\bar{t}_2} + 2\sum_{j=0}^{N}(-1)^{N+j}|\hat{N}_j, 2j+1;\ \widehat{M-2}, M+1, M| \\
&\qquad - 2|\widehat{N-1}, 2N+2;\ \widehat{M-1}, M+1| \\
&\qquad + 2\sum_{j=0}^{N}(-1)^{N+j}|\hat{N}_j, 2j+1;\ \widehat{M-1}, M+2|.
\end{aligned}$$

So from the identity

$$\left(\sum_{i=0}^{N+M+2}\lambda_i\right)\left(\sum_{i=0}^{N+M+2}\lambda_i^2\right)\tau\cdot\tau = \left(\sum_{i=0}^{N+M+2}\lambda_i\right)\tau\cdot\left(\sum_{i=0}^{N+M+2}\lambda_i^2\right)\tau$$

we have

$$\begin{aligned}
\tfrac{1}{2}(\tau_{\bar{t}_1\bar{t}_2}\tau - \tau_{\bar{t}_1}\tau_{\bar{t}_2}) = \Big\{ &|\widehat{N-1}, 2N+2;\ \widehat{M-1}, M+1| \\
&- \sum_{j=0}^{N}(-1)^{N+j}|\hat{N}_j, 2j+1;\ \widehat{M-2}, M+1, M|
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{j=0}^{N}(-1)^{N+j}|\hat{N}_j, 2j+1;\ \widehat{M-1}, M+2|\Big\}|\hat{N};\ \hat{M}| \\
&+ \Big\{|\hat{N};\ \widehat{M-2}, M+1, M| + |\hat{N}, \widehat{M-1}, \widehat{M+2}|\Big\} \\
&\times \sum_{j=0}^{N}(-1)^{N+j}|\hat{N}_j, 2j+1;\ \hat{M}| \\
&- |\hat{N};\ \widehat{M-1}, M+1||\widehat{N-1}, 2N+2;\ \hat{M}|.
\end{aligned}
$$

By using the identity (2.8) we have

$$
\begin{aligned}
\tau_{\bar{t}_1\bar{t}_2}\tau - \tau_{\bar{t}_1}\tau_{\bar{t}_2} - 4\rho\sigma = {} & 2\sum_{j=0}^{N}(-1)^{N+j}|\hat{N}_j;\ \widehat{M+1}||\hat{N}, 2j+1;\ \widehat{M-2}, M| \\
&- 2|\widehat{N+1};\ \widehat{M-1}||\widehat{N-1};\ \widehat{M+1}| \\
&- 2\sum_{j=0}^{N}(-1)^{N+j}|\hat{N}_j;\ \hat{M}, M+2||\hat{N}, 2k+1;\ \widehat{M-1}|.
\end{aligned}
$$

For convenience, we denote the right-hand side of the above equality by $2S$. By using identity (2.7) we have

$$
\begin{aligned}
|\widehat{N+1};\ \widehat{M-1}| = {} & \left(\sum_{i=0}^{N+M+2}\lambda_i\right)|\hat{N}, 2N+1;\ \widehat{M-1}| + |\hat{N}, 2N+1;\ \widehat{M-2}, M| \\
&- \sum_{j=0}^{N-1}(-1)^{N+j}|\hat{N}_j, 2j+1, 2N+1;\ \widehat{M-1}|
\end{aligned}
$$

so

$$
\begin{aligned}
|\widehat{N+1};\ \widehat{M-1}||\widehat{N-1};\ \widehat{M+1}| = {} & |\hat{N}, 2N+1;\ \widehat{M-2}, M||\widehat{N-1};\ \widehat{M+1}| \\
&- |\hat{N}, 2N+1;\ \widehat{M-1}||\widehat{N-1};\ \hat{M}, M+2| \\
&- \sum_{j=0}^{N-1}(-1)^{N+j}|\hat{N}_j, 2j+1, 2N+1;\ \widehat{M-1}||\widehat{N-1};\ \widehat{M+1}| \\
&+ \sum_{j=0}^{N-1}(-1)^{N+j-1}|(\widehat{N-1})_j, 2j+1;\ \widehat{M+1}||\hat{N}, 2N+1;\ \widehat{M-1}|
\end{aligned}
$$

thus we have

$$
\begin{aligned}
S = {} & \sum_{j=0}^{N-1}(-1)^{N+j}|\hat{N}_j;\ \widehat{M+1}||\hat{N}, 2j+1;\ \widehat{M-2}, M| \\
&- \sum_{j=0}^{N-1}(-1)^{N+j}|\hat{N}_j;\ \hat{M}, M+2||\hat{N}, 2j+1;\ \widehat{M-1}| \\
&+ \sum_{j=0}^{N-1}(-1)^{N+j}|\hat{N}_j, 2j+1, 2N+1;\ \widehat{M-1}||\widehat{N-1};\ \widehat{M+1}| \\
&+ \sum_{j=0}^{N-1}(-1)^{N+j}|(\widehat{N-1})_j, 2j+1;\ \widehat{M+1}||\hat{N}, 2N+1;\ \widehat{M-1}|.
\end{aligned}
$$

By using the equality

$$|\hat{N}_j;\ \hat{M}, M+2| = -\left(\sum_{i=0}^{N+M+2} \lambda_i\right) |\hat{N}_j;\ \hat{M}, M+1| + \left\{\sum_{l<j}(-1)^{N+1+l} + \sum_{l>j}(-1)^{N+l}\right\} |\hat{N}_{jl}, 2l+1;\ \hat{M}, M+1|$$

we can rewrite S as follows:

$$\begin{aligned}
S = &\sum_{j=0}^{N-1}(-1)^{N+j}|\hat{N}_j;\ \widehat{M+1}||\hat{N}, 2j+1; \widehat{M-2}, M| \\
&+ \sum_{j=0}^{N-1}(-1)^{N+j}\Big\{(-1)^{N+l}\sum_{l\neq j}^{N}|\hat{N}_l, 2l+1, 2j+1;\ \widehat{M-1}| \\
&+ |\hat{N}, 2j+2;\ \widehat{M-1}| - |\hat{N}, 2j+1;\ \widehat{M-2}, M|\Big\}|\hat{N}_j;\ \widehat{M+1}| \\
&- \sum_{j=0}^{N-1}(-1)^{N+j}|\hat{N}, 2j+1;\ \widehat{M-1}| \left\{\sum_{l<j}(-1)^{N+1+l} + \sum_{l>j}(-1)^{N+l}\right\} \\
&\times |\hat{N}_{jl}, 2l+1;\ \widehat{M+1}| \\
&+ \sum_{j=0}^{N-1}(-1)^{N+j}|\hat{N}_j, 2j+1, 2N+1;\ \widehat{M-1}||\widehat{N-1};\ \widehat{M+1}| \\
&+ \sum_{j=0}^{N-1}(-1)^{N+j}|(\widehat{N-1})_j, 2j+1;\ \widehat{M+1}||\hat{N}, 2N+1;\ \widehat{M-1}|.
\end{aligned}$$

To prove $S=0$, we need the following identities:

$$\det\begin{pmatrix} \widehat{N-1}, & \hat{N}_j, & 2j+1, & 2N+1; & \widehat{M+1}, & 0 \\ \widehat{N-1}, & \hat{N}_j, & 2j+1, & 2N+1; & 0, & \widehat{M-1} \end{pmatrix} = 0 \tag{3.5}$$

$$\det\begin{pmatrix} \hat{N}_j, & \hat{N}_l, & 2l+1, & 2j+1; & \widehat{M+1}, & 0 \\ \hat{N}_j, & \hat{N}_l, & 2l+1, & 2j+1; & 0, & \widehat{M-1} \end{pmatrix} = 0. \tag{3.6}$$

they can be equivalently written as

$$\begin{aligned}
&|\hat{N}_j, 2j+1, 2N+1;\ \widehat{M-1}||\widehat{N-1};\ \widehat{M+1}| + |(\widehat{N-1})_j, 2j+1;\ \widehat{M+1}| \\
&\quad\times |\hat{N}, 2N+1;\ \widehat{M-1}| + |\widehat{N-1}, 2N+1, 2j+1;\ \widehat{M-1}||\hat{N}_j;\ \widehat{M+1}| \\
&\quad- |\hat{N}, 2j+1;\ \widehat{M-1}||(\widehat{N-1})_j, 2N+1;\ \widehat{M+1}| \\
&= 0 \qquad 0 \leqslant j \leqslant N-1
\end{aligned} \tag{3.7}$$

$$\begin{aligned}
&|\hat{N}_l, 2l+1, 2j+1;\ \widehat{M-1}||\hat{N}_j;\ \widehat{M+1}| + |\hat{N}_{jl}, 2l+1;\ \widehat{M+1}| \\
&\quad\times |\hat{N}, 2j+1;\ \widehat{M-1}| + |\hat{N}_j, 2j+1, 2l+1;\ \widehat{M-1}||\hat{N}_l;\ \widehat{M+1}| \\
&\quad- |\hat{N}_{lj}, 2j+1;\ \widehat{M+1}||\hat{N}, 2l+1;\ \widehat{M-1}| \\
&= 0 \qquad 0 \leqslant l < j \leqslant N-1.
\end{aligned} \tag{3.8}$$

Since when $j \leqslant N-1$, $|\hat{N}, 2j+2;\ \widehat{M-1}| = 0$, from identities (3.7) and (3.8) we know that $S = 0$. Thus we have proved that τ, ρ, σ satisfy equation (3.2*a*). Similarly, we can prove that τ, ρ, σ also satisfy equations (3.2*b*) and (3.2*c*). The theorem is proved.

4. A conjecture on the Wronskian structure for the solutions od the k-constrained KP hierarchy

In the preceding section, we have shown that the Yajima–Oikawa equation (1.6) which is the first flow of the 2-constrained KP hierarchy, has solutions in generalized double Wronskian form. In this section, we shall give a conjecture on the Wronskian structure for the solutions of the k-constrained KP hierarchy. For this purpose, we first give the bilinear equations for the k-constrained KP hierarchy. From [7, 11] we know that if we define τ, ρ, σ as follows:

$$\frac{\partial^2}{\partial t_1 \partial t_2} \ln \tau = \mathrm{Res}_\partial L^n \qquad q = \frac{\rho}{\tau} \qquad r = \frac{\sigma}{\tau}$$

where $\mathrm{Res}_\partial L^n$ denotes the coefficient of ∂^{-1} of the micro-differential operator L^n, then the k-constrained KP hierarchy can be put into the following bilinear equations:

$$\exp\left(\sum_{i=1}^{\infty} y_i D_i\right) \sigma \cdot \rho = \sum_{j=0}^{\infty} p_j(-2y) p_{j+k+1}(\tilde{D}) \exp\left(\sum_{i=1}^{\infty} y_i D_i\right) \tau \cdot \tau \tag{4.1a}$$

$$\exp\left(\sum_{i=1}^{\infty} y_i D_i\right) \tau \cdot \rho = \sum_{j=0}^{\infty} p_j(-2y) p_j(\tilde{D}) \exp\left(\sum_{i=1}^{\infty} y_i D_i\right) \rho \cdot \tau \tag{4.1b}$$

$$\exp\left(\sum_{i=1}^{\infty} y_i D_i\right) \sigma \cdot \tau = \sum_{j=0}^{\infty} p_j(-2y) p_j(\tilde{D}) \exp\left(\sum_{i=1}^{\infty} y_i D_i\right) \tau \cdot \sigma \tag{4.1c}$$

where $y = (y_1, y_2, \cdots)$, $\tilde{D} = (D_1, \frac{1}{2}D_2, \frac{1}{3}D_3, \cdots)$, D_n, $n = 1, 2, \cdots$ is the Hirots's bilinear operator with respect to t_n and $p_j(t)$, $j = 1, 2, \cdots$ is the Schur polynomial defined by

$$e^{\xi(t,\lambda)} = \sum_{j=0}^{\infty} p_j(t)\lambda^j \qquad \xi(t, \lambda) = \sum_{i=1}^{\infty} \lambda^i t_i. \tag{4.2}$$

We now give the following conjecture:

Conjecture. Let

$$\tau = \det\left(\varphi, \varphi^{(k)}, \cdots, \varphi^{(kN)}, \psi, \psi^{(1)}, \cdots, \psi^{(M)}\right) \tag{4.3a}$$

$$\rho = \det\left(\varphi, \varphi^{(k)}, \cdots, \varphi^{(kN+k)}, \psi, \psi^{(1)}, \cdots, \psi^{(M-1)}\right) \tag{4.3b}$$

$$\sigma = \det\left(\varphi, \varphi^{(k)}, \cdots, \varphi^{(kN-k)}, \psi, \psi^{(1)}, \cdots, \psi^{(M+1)}\right) \tag{4.3c}$$

where $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_{N+M+2})^T$, $\psi = (\psi_1, \psi_2, \cdots, \psi_{N+M+2})^T$, $\varphi_i = a_i e^{\xi(\vec{t},\lambda_i)}$, $\psi_i = b_i e^{-\xi(\vec{t},\lambda_i)}$, and a_i, b_i, λ_i are arbitrary constants, M, N are arbitrary positive integers. Then τ, ρ, σ defined by (4.3) satisfy equations in (4.1), thus giving a solution of the k-constrained KP hierarchy.

From the Wronskian solutions of the AKNS hierarchy given in [10] we know that the conjecture is verified for the 1-constrained KP hierarchy, and for the first flow of the 2-constrained KP hierarchy the conjecture is verified by the result of section 3. For general k, it can be shown that the double-soliton solution of the k-constrained KP hierarchy found in [8] can be put into the form given in the conjecture with $M = N = 2$.

5. Conclusion

We have proved that the first flow of the 2-constrained KP hierarchy has solution in generalized double Wronskian form, and then based on this fact and the double Wronskian solutions of the 1-constrained KP hierarchy, the double-soliton solution of the k-constrained KP hierarchy, we give a conjecture that the k-constrained KP hierarchy has solutions in generalized double Wronskian form. We note that if our conjecture is true for general k, N, M, then the τ defined by (4.3a) is also a tau-function for the KP hierarchy, and it may be put into the form of a usual Wronskian determinant [1, 11, 12], this provides us a hint to verify our conjecture. On the other hand, from [8] we know that solutions τ, ρ, σ of the k-constrained KP hierarchy can be obtained by using the free Fermion operators, and they can be expressed by vertex operators, thus another possible way to see whether our conjecture is true is to relate the τ, ρ, σ defined by (4.3) with the τ, ρ, σ expressed by the vertex operators given in [8].

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